Discrete systems related to some equations of the Painlevé-Gambier classification

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Abstract

We derive integrable discrete systems which are contiguity relations of two equations in the Painlevé-Gambier classification depending on some parameter. These studies extend earlier work where the contiguity relations for the six transcendental Painlevé equations were obtained. In the case of the Gambier equation we give the contiguity relations for both the continuous and the discrete system.

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1. Introduction

The relation of discrete Painlevé equations to their, better known, continuous analogues has by now been firmly established. As a matter of fact one of the very first methods proposed to the derivation of discrete Painlevé equations (d-$\mathbb{P}$) has been the one based on the auto-Bäcklund/Schlesinger transformations of continuous Painlevé equations [1]. (This link goes even further back in time since one of the first discrete Painlevé equation obtained, even before the notion of a d-$\mathbb{P}$ was explicited, was the one derived by Jimbo and Miwa [2] as a contiguity relation of the solutions of $\mathrm{P_{II}}$). The general method for this derivation has been amply explained in previous publications [3]. One uses the auto-Bäcklund/Schlesinger transformations in order to derive relations for the solutions of a continuous Painlevé equations for contiguous values of the one of its parameters and one gets precisely a d-$\mathbb{P}$. The discrete equation related to $\mathrm{P_{II}}$ has been first obtained, as explained above, by Jimbo and Miwa [2] and rediscovered in [1]. The d-$\mathbb{P}$'s related to $\mathrm{P_{III}}$ were first given in [1] and have been recently reexamined in detail in [4]. The discrete equation related to $\mathrm{P_{IV}}$ was first obtained in [1] and was again studied in [5]. Some d-$\mathbb{P}$'s related to $\mathrm{P_{V}}$ were first given in [1] but the relation of $\mathrm{P_{V}}$ to asymmetric d-$\mathrm{P_{II}}$ is under preparation [6]. Finally the discrete systems obtained from the auto-Bäcklund of $\mathrm{P_{VI}}$ were presented in full detail in [7]. Thus the discrete, difference, Painlevé equations can be interpreted as the contiguity relations of the continuous ones. (This would tend to cast a doubt on the fundamental character of the d-$\mathbb{P}$'s. While this is justified as far as the difference d-$\mathbb{P}$'s are concerned there exists a second type of d-$\mathbb{P}$'s which have no relation to continuous Painlevé equations whatsoever. These are the multiplicative equations, q-$\mathbb{P}$'s [8] which are purely discrete objects and thus of the most fundamental nature). One of the consequences of this relation between continuous and discrete Painlevé equations is that one can obtain the Lax pairs of the latter from the mere existence of the Lax pairs and Schlesinger transformations of the former. The only complication comes from the fact that given a d-$\mathbb{P}$ one has to identify the procedure for its derivation based on the Schlesinger transformation of some continuous Painlevé equation. This can be done in the frame of the geometric description of Painlevé equations which was introduced in [9] and dubbed the "Grand Scheme". This description based, on the geometry of the affine Weyl group weight spaces, has turned out to be a most valuable tool for the classification of the d-$\mathbb{P}$'s, which minimised their uncontrollable proliferation.

In the approach we have described above the emphasis was on the discrete analogues of the six transcendental equations of Painlevé. As a matter of fact the remaining integrable ODE's of the Painlevé-Gambier classification have received little attention. Our current knowledge is essentially limited to the discrete analogues of the 24 basic equations of Gambier [10]. In particular no results exist on the contiguity relations other than for the six equations

of Painlevé. In this paper we intend to focus on just this question and present a first, exploratory, study of some discrete equations related to continuous systems which are not one of the six Painlevé equations. In particular we shall obtain the mappings related to the equations 35 and 27 of the Painlevé-Gambier classification. Equation 27 is also known as the Gambier equation and has been studied in detail in [11], where its discrete equivalent has been obtained. Since, as was shown in [12], both the continuous and discrete Gambier equations possess Schlesinger transformations, one can obtain the contiguity relations not only of the continuous but also of the discrete system. This result is presented in the last part of the present paper.

2. Discrete systems related to P_{35}

The first system we are going to examine is the equation #35 in the Painlevé-Gambier classification [13]:

$$w'' = \frac{2}{3w}w'^2 - \left(\frac{2}{3}w - \frac{2}{3}q - \frac{r}{w}\right)w' + \frac{2}{3}w^3 - \frac{10}{3}qw^2 + \left(4q' + r + \frac{8}{3}q^2\right)w + 2qr - 3r' - \frac{3r^2}{w} \tag{2.1}$$

with $r = -z/3 - 2(q' + q^2)/3$ and q one particular solution of P_{II},

$$v'' = 2v^3 + zv + \alpha, \tag{2.2}$$

where z is the independent variable and α is a constant. The way to integrate this equation is based on the following Miura transformation. We start with the *general* solution of P_{II} v, and we construct the quantity

$$w = \frac{v' - q'}{v - q} + v + q \tag{2.3}$$

which solves (2.1). Moreover, starting from the general solution of (2.1) w, the quantity

$$v = \frac{1}{3w}(w' + w^2 - qw - 2q' - 2q^2 - z) \tag{2.4}$$

solves P_{II}. Equations (2.3) and (2.4) define a Miura transformation relating P_{II} and P_{35}. Another such Miura relation between P_{II} and some other equation of the Painlevé-Gambier list is already known, that relating P_{II} and P_{34}. It has been studied in detail in [14] in both the continuous and the discrete case.

Equation P_{35} has a parameter, α, through its relation to P_{II}. Let us write w_α for the solution of (2.1) corresponding to the parameter α and v_α and q_α for the solutions of P_{II} entering in (2.1), (2.3) and (2.4). Next we consider the auto-Bäcklund transformations of P_{II} [15]:

$$v_{\alpha+1} = -v_\alpha - \frac{\alpha + 1/2}{v'_\alpha + {v_\alpha}^2 + z/2}, \tag{2.5a}$$

$$v_{\alpha-1} = -v_\alpha + \frac{\alpha - 1/2}{v'_\alpha - {v_\alpha}^2 - z/2}, \tag{2.5b}$$

corresponding to solutions where the parameter α varies by integer quantities. Thus, following the general theory presented in [1] one can introduce a discrete system associated to these transformations as a mapping which is the contiguity relation for the solutions of $\mathrm{P_{II}}$. This equation was in fact first obtained in [2] and is now known (at least to the present authors) under the name of alternate d-$\mathrm{P_I}$. It has the form

$$\frac{\alpha - 1/2}{v_{\alpha-1} + v_\alpha} + \frac{\alpha + 1/2}{v_{\alpha+1} + v_\alpha} = -2{v_\alpha}^2 - z. \tag{2.6}$$

This is the equation satisfied by all solutions of $\mathrm{P_{II}}$, and in particular by q, when we vary the parameter of $\mathrm{P_{II}}$ using (2.5).

Using the Miura (2.3) we can obtain the solutions of $\mathrm{P_{35}}$, $w_{\alpha+1}$ and $w_{\alpha-1}$, corresponding to the parameter values $\alpha \pm 1$:

$$w_{\alpha+1} = \frac{v'_{\alpha+1} - q'_{\alpha+1}}{v_{\alpha+1} - q_{\alpha+1}} + v_{\alpha+1} + q_{\alpha+1}, \tag{2.7a}$$

$$w_{\alpha-1} = \frac{v'_{\alpha-1} - q'_{\alpha-1}}{v_{\alpha-1} - q_{\alpha-1}} + v_{\alpha-1} + q_{\alpha-1}. \tag{2.7b}$$

Using (2.2), (2.3), (2.5) and (2.7), one can write an equation relating $w_{\alpha+1}$, w_α and $w_{\alpha-1}$ alone where neither v nor its derivatives appear. Interpreting α as a discrete independent variable, we obtain thus a second-order discrete equation for w. This mapping is however quadratic in $w_{\alpha+1}$ and $w_{\alpha-1}$ and thus (as was also argued in [16]) cannot be integrable. Our argument is based on the fact that the evolution of such a mapping leads in general to an exponential number of images, and preimages, of the initial point. The non-integrability of this non-singlevalued system is not in contradiction to the fact that we can obtain one solution of this mapping, namely the one furnished by the evolution of the system before elimination. This solution is the only one that we know how to describe, while a system with exponentially increasing number of branches eludes a full description.

In order to proceed, instead of working with a second-order mapping, we choose to derive a system of two coupled equations. We first make use of the discrete symmetry of the solutions of $\mathrm{P_{II}}$: if v_α is a solution of $\mathrm{P_{II}}$ with parameter α, then $-v_\alpha$ is a solution of $\mathrm{P_{II}}$ with parameter $-\alpha$. We then introduce the quantity

$$u_\alpha = \frac{v'_\alpha - q'_\alpha}{v_\alpha - q_\alpha} - v_\alpha - q_\alpha \tag{2.8}$$

which is obtained from the expression (2.3) for w_α after implementing the reflections $v_\alpha \to -v_\alpha$ and $q_\alpha \to -q_\alpha$. The quantity u_α is thus the solution of a *slightly modified* $\mathrm{P_{35}}$ equation (with $-q_\alpha$ instead of q_α) that we shall denote $\tilde{\mathrm{P}}_{35}$. Note that, from (2.3) and (2.8) we have:

$$w_\alpha - u_\alpha = 2(v_\alpha + q_\alpha). \tag{2.9}$$

We start by writing an equation between $w_{\alpha-1}$, w_α and u_α only, using (2.2), (2.3), (2.5), (2.7) and (2.9). Similarly we also obtain an equation relating $u_{\alpha+1}$, u_α and w_α. We thus find the second-order discrete system:

$$u_{\alpha+1} = w_\alpha + \frac{2(q_{\alpha+1}+q_\alpha)^2 w_\alpha^2}{(q_{\alpha+1}+q_\alpha)(u_\alpha w_\alpha - w_\alpha^2 + 2(q_\alpha - q_{\alpha+1})w_\alpha) + 2\alpha + 1}, \tag{2.10a}$$

$$w_\alpha = u_\alpha + \frac{1-2\alpha}{u_\alpha(q_\alpha + q_{\alpha-1})} + \frac{2w_{\alpha-1}(q_\alpha - q_{\alpha-1}) - 4q_\alpha u_\alpha}{w_{\alpha-1} - u_\alpha}, \tag{2.10b}$$

where q_α is a particular solution of (2.6).

This system introduces the contiguity relations for equation P_{35} (2.1) and $\tilde{\mathrm{P}}_{35}$. It can be integrated with the help of alternate d-$\mathrm{P_I}$, equation (2.6). We first write v'_α in terms of $v_{\alpha+1}$ and v_α and q'_α in terms of $q_{\alpha+1}$ and q_α using the Schlesinger transformation (2.5). Using these expressions in the definition (2.3), we obtain the following:

$$w_\alpha = \frac{\alpha + 1/2}{v_\alpha - q_\alpha}\left[\frac{1}{q_\alpha + q_{\alpha+1}} - \frac{1}{v_\alpha + v_{\alpha+1}}\right]. \tag{2.11}$$

The quantity u_α is obtained in terms of v_α and w_α using (2.9). Thus, once v_α is known, we can construct w_α and u_α in a straigthforward way.

At this point we can remark that while (2.10) is integrable by construction, the condition for integrability, namely that q is a specific function, may appear somewhat contrived. In what follows we shall show that the fact that q satisfies alternate d-$\mathrm{P_I}$ can be obtained precisely through the singularity confinement criterion. We start by writing (2.10) and (2.6) in a more convenient way in order to implement the singularity analysis.:

$$u_{\alpha+1} = w_\alpha + \frac{2(q_{\alpha+1}+q_\alpha)^2 w_\alpha^2}{(q_{\alpha+1}+q_\alpha)(w_\alpha u_\alpha - w_\alpha^2 + 2(q_\alpha - q_{\alpha+1})w_\alpha) + 2z_{\alpha+1}}, \tag{2.12a}$$

$$w_\alpha = u_\alpha - \frac{2z_\alpha}{u_\alpha(q_\alpha + q_{\alpha-1})} + \frac{2w_{\alpha-1}(q_\alpha - q_{\alpha-1}) - 4q_\alpha u_\alpha}{w_{\alpha-1} - u_\alpha}, \tag{2.12b}$$

where q and z are two functions to be determined.

There are three possible sources of singularities for (2.12): $u_\alpha = 0$, $w_\alpha = 0$ and the case when the denominator in the rhs of equation (2.12a) happens to be 0 for some value of α. We first examine the singularity $u_\alpha = 0$. Let the initial condition $w_{\alpha-1}$ be free and introduce the small parameter ϵ: $u_\alpha = \epsilon$. At the lowest order in ϵ we find, $w_\alpha = 1/\epsilon + \ldots$, $u_{\alpha+1} = 1/\epsilon + \ldots$, $w_{\alpha+1} = \mathcal{O}(\epsilon)$. Then $u_{\alpha+2}$ must take the indefinite form 0/0 in order to contain the information on $w_{\alpha-1}$ when $\epsilon \to 0$. Implementing this condition, we find that q_α must indeed satisfy

$$\frac{z_\alpha}{q_{\alpha-1} + q_\alpha} + \frac{z_{\alpha+1}}{q_{\alpha+1} + q_\alpha} = -2{q_\alpha}^2 - \gamma, \tag{2.13}$$

for some constant γ and z_α must be linear in α. (The two other singularities are automatically confined for all functions q_α and z_α.) Thus the singularity confinement criterion gives for q_α and z_α precisely the constraints expected from the construction of (2.12) from P_{35}.

3. A mapping related to the continuous Gambier equation

The second discrete system we are going to derive is the contiguity relation of the solutions of the Gambier equation. The continuous Gambier system can be written as a cascade of two Riccati equations:

$$y' = -y^2 + c, \tag{3.1a}$$

$$x' = ax^2 + nxy + \sigma, \tag{3.1b}$$

where a, c and σ are free functions of the independent variable z and n is an integer. By a change of variable, the function σ can be scaled to 0 or 1 but, for the purpose of constructing Schlesinger transformations, it is more convenient to keep it as a free function. Moreover, if we require that equation (3.1) possess the Painlevé property, the functions will be subject to one more constraint. This constraint depends on the value of the integer parameter n.

Let us write x_n, a_n and σ_n, the quantities appearing in (3.1) corresponding to the coupling parameter n in (3.1b). Then, we can introduce the following Schlesinger transformation [12,17]. It relates the solutions of (3.1) to that of an equation of the same form but with a coupling parameter $\tilde{n} = n + 2$. This transformation is given by:

$$x_{n+2} = -\frac{\sigma_{n+2}}{n+1}\left[y + \frac{1}{n+2}\frac{a_n'}{a_n} + \frac{a_n x_n}{n+1}\right]^{-1}, \tag{3.2a}$$

$$\frac{\sigma_{n+2}'}{\sigma_{n+2}} = -\frac{n}{n+2}\frac{a_n'}{a_n}, \tag{3.2b}$$

$$a_{n+2} = \frac{n+1}{\sigma_{n+2}}\left(c_n + \frac{a_n\sigma_n}{n+1} + \frac{1}{n+2}\frac{a_n''}{a_n} - \frac{n+3}{(n+2)^2}\frac{a_n'^2}{a_n^2}\right). \tag{3.2c}$$

The inverse of this transformation is given by

$$x_{n-2} = \frac{1-n}{a_{n-2}}\left[\frac{\sigma_n}{x_n(n-1)} + y + \frac{1}{2-n}\frac{\sigma_n'}{\sigma_n}\right], \tag{3.3a}$$

$$\frac{a_{n-2}'}{a_{n-2}} = -\frac{n}{n-2}\frac{\sigma_n'}{\sigma_n}, \tag{3.3b}$$

$$\sigma_{n-2} = \frac{1-n}{a_{n-2}}\left(c_n + \frac{a_n\sigma_n}{1-n} + \frac{1}{2-n}\frac{\sigma_n''}{\sigma_n} + \frac{n-3}{(n-2)^2}\frac{\sigma_n'^2}{\sigma_n^2}\right). \tag{3.3c}$$

We can obtain a second-order mapping for x eliminating y between equations (3.2a) and (3.3a).

$$x_{n-2}\frac{a_{n-2}}{n-1} - x_n\frac{a_n}{n+1} - \frac{\sigma_n'}{\sigma_n}\frac{1}{n-2} - \frac{a_n'}{a_n}\frac{1}{n+2} + \frac{1}{x_n}\frac{\sigma_n}{n-1} - \frac{1}{x_{n+2}}\frac{\sigma_{n+2}}{n+1} = 0 \tag{3.4}$$

There is another way to present equation (3.4) if we write it as a system, involving explicitly y and using the fact that it does not depend on n. The two equations (3.2b) and (3.2c) are equivalent to the upshifts (by 2 units of n) of (3.3b) and (3.3c). Equation (3.2b) is the compatibility condition between (3.2a) and the upshift of (3.3a).

We obtain thus

$$y_{n+2} = y_n \tag{3.5a}$$

$$x_{n-2} = \frac{1-n}{a_{n-2}}\left[\frac{\sigma_n}{x_n(n-1)} + y_n + \frac{1}{2-n}\frac{\sigma'_n}{\sigma_n}\right]. \tag{3.5b}$$

The evolution of σ and a is obtained through equations (3.2b)-(3.2c). System (3.5) is just a particular case of the Gambier mapping introduced in [18]. It is remarkable that the contiguity relation of the solutions of the Gambier equations with respect to the one integer parameter, namely n, is again a (discrete) Gambier equation. However the latter does not have the full freedom of the Gambier mapping (in perfect parallel to the relation between continous Painlevé equations and d-P's.)

4. A mapping related to the discrete Gambier equation

In this section we shall concentrate on the generic discrete version of the Gambier equation. As was shown in [12] it can be written as a system of two discrete Riccati's in cascade:

$$y_{n+1} = \frac{y_n + c_n}{y_n + 1}, \tag{4.1a}$$

$$x_{n+1} = \frac{x_n(y_n - r_n) + q_n(y_n - s_n)}{x_n y_n}. \tag{4.1b}$$

(Other choices for the Gambier mapping do exist but equation (4.1) is the most convenient for the derivation of the Schlesinger transformation.) The mapping was analysed in [17] using the singularity confinement method. The only singularity that plays a role is that induced by a special value of y, which, given the form of (4.1), is $y = 0$. The way for this singularity to be confined is to require that the value of x go through an indeterminate value $0/0$ after a certain number of steps, say N. This means in particular that x must be zero, while s appearing in (4.1b) must be equal to the quantity ψ_N (introduced in [12]) which is the Nth iterate of $y = 0$ under (4.1a), N times downshifted i.e. $\psi_{0,n} = 0$, $\psi_{1,n} = c_{n-1}$, $\psi_{2,n} = \frac{c_{n-2}+c_{n-1}}{c_{n-2}+1}$, etc. As was shown in [12], at the continuous limit, the number N of steps, necessary for confinement, goes over to the parameter n appearing in the continuous Gambier equation (3.1).

The discrete version of the Schlesinger transformation (3.2) is a transformation which relates the solution of a mapping confining in N steps to that of a mapping confining in $N+2$ steps.

The complete analysis for the derivation of the discrete Schlesinger can be found in [12]. Here we shall just state the result:

$$\tilde{x}_{n+1} = \phi_n \frac{x_n r_n + q_n \psi_{N,n}}{y_n} \frac{y_n - \psi_{N+1,n}}{\psi_{N,n-1}(x_n - 1) + r_{n-1}}, \tag{4.2}$$

where $\tilde{x}$ is the solution requiring $N+2$ steps for confinement and x stands for the solution necessitating just N steps. The quantity ϕ is given by

$$\phi_n = \frac{(\psi_{N,n-1}+1)(\psi_{N,n-1}\psi_{N,n-2}q_{n-1} + \psi_{N,n-2}r_{n-1} - r_{n-1}\, r_{n-2})}{(\psi_{N,n-2} - r_{n-2})(\psi_{N,n}q_n + r_n r_{n-1} + r_n) + r_n q_{n-1}\, \psi_{N,n-2}(1+\psi_{N,n-1})}. \tag{4.3}$$

With this proper choice of ϕ we find that $\tilde{x}$ satisfies an equation:

$$\tilde{x}_{n+1} = \frac{\tilde{x}_n(y_n - \tilde{r}_n) + \tilde{q}_n(y_n - \psi_{N+2,n})}{\tilde{x}_n y_n}, \tag{4.4}$$

where $\tilde{r}$ and $\tilde{q}$ are functions completely determined in terms of q, r and c. In a similar way we can derive the Schlesinger transformation invoving a solution which takes $N-2$ steps to confine. We find:

$$\underset{\sim}{x}_n = \omega_n \frac{x_n(y_n - c_{n-1}) + g_n(y_n - \psi_{N,n})}{x_n(y_n - c_{n-1}) + h_n(y_n - \psi_{N,n})}, \tag{4.5}$$

where

$$g_n = \frac{q_{n+1}(\psi_{N-1,n+1} - c_n)}{\psi_{N-1,n+1} - r_{n+1}}, \tag{4.6a}$$

$$h_n = \frac{(c_{n-1} - r_n)(c_n - 1)}{c_{n-1} + c_n - \psi_{N,n+1}(c_{n-1}+1)}, \tag{4.6b}$$

$$\omega_n = \frac{u_n(1 - c_n) + h_n(1 + c_n - 2\psi_{N,n+1})}{u_n(1 - c_n) + g_n(1 + c_n - 2\psi_{N,n+1})}, \tag{4.6c}$$

$$u_n = 1 - r_n + \frac{q_n(c_{n-1} - 1)}{h_{n-1}}. \tag{4.6d}$$

We can now obtain the contiguity relation of the solutions of the Gambier mapping by eliminating y between (4.2) and (4.5).

On the other hand, just as in the case of the continuous Gambier system there exists a simpler way to write this mapping. The value of y appearing in (4.2) or (4.5) is independent of N. Thus a far simpler way to present the mapping is for instance as

$$\underset{\sim}{y}_n = y_n, \tag{4.7a}$$

$$\underset{\sim}{x}_n = \omega_n \frac{x_n(y_n - c_{n-1}) + g_n(y_n - \psi_{N,n})}{x_n(y_n - c_{n-1}) + h_n(y_n - \psi_{N,n})}. \tag{4.7b}$$

Again we remark that (4.7) is a mapping of Gambier type although of a very particular one.

5. Conclusion

In this paper we have studied the contiguity relations of the solutions of two equations in the Painlevé-Gambier classification which contain some parameter. This kind of investigation was previously limited to the six transcendental equations of Painlevé. The main result of those studies was to show that the discrete Painlevé equations are just the contiguity relations of the continuous Painlevé equations. Here we have extended this approach to equations which, while not being one of the six Painlevé equations are integrable and, in fact, belong to the same classification. In the case of the Gambier equation we have used the Schlesinger transformation which is associated to the integer parameter n appearing in the equation. Our result in the case of the continuous Gambier equation is that the contiguity relation of the solutions satisfies a mapping which assumes the form of a discrete analogue of the Gambier system, in a nice analogy to the situation for the continuous and discrete Painlevé equations. In parallel to our approach for d-$\mathbb{P}$'s the discrete Gambier system has also been analysed. We have shown that using the Schlesinger transformation that also exist in this case one can derive the contiguity relation of the solutions which turns out to be in the form of (a special case of) a Gambier mapping. We expect our approach to be applicable to other equations of the Painlevé-Gambier classification which contain parameters and for which Miura/Schlesinger transformations exist. Now that the question of classification of discrete Painlevé equations is in the process of being settled [19,20] it is interesting to analyse the remaining equations in the classification and work out their implications in the discrete case.

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